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Planetary semi-geostrophic equations derived from Hamilton's principle

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A new set of balanced equations (to be called planetary semi-geostrophic equations) for planetary-scale flow is derived from Hamilton's principle and constitutes a natural generalization of the semi-geostrophic equations of motion. Analogues of the global conservation of energy and the Lagrangian conservation of potential vorticity follow automatically by introducing approximations directly into the Hamiltonian in such a way that time and particle label symmetries are preserved. Two approximations are required: first, the kinetic energy associated with the component of velocity parallel to the axis of rotation is neglected; and secondly, the Lagrangian rate of change of the wind and pressure gradient directions (when projected onto the equatorial plane) must be small compared with twice the angular rotation rate of the system. Although the first of these approximations entails some loss of accuracy for application to the terrestrial atmosphere it is not nearly as severe as that for the Phillips type II geostrophic equations in which all of the kinetic energy is omitted from the Hamiltonian. The resulting equations take exactly the same form as the f-plane semi-geostrophic equations apart from a modification to the pseudo-density appearing in the continuity equation. They are also amenable to the geostrophic momentum coordinate transformation – a device which has had considerable impact on the theory of atmospheric fronts. In order to assess the accuracy of the equations, three different linearized eigenvalue problems on the sphere are solved and compared with the equivalent primitive equation problems. Eigenmodes are least accurate for high-zonal-wavenumber disturbances with grave meridional structure. Stationary, baroclinic planetary waves with zonal wavenumber less than ≈ 7 are shown to be accurately treated. The equations also support equatorially trapped Kelvin and Rossby modes which are accurate in the long-wave limit for meteorologically relevant equivalent depths.

1. Introduction

Arguably the greatest achievement of meteorological science to date has been the development and operational use of global numerical models to forecast the weather (see e.g. White *et al.* 1987). Although parametrization of subgrid-scale physical processes such as radiative transfer and boundary-layer momentum transport render the mathematical problem extremely complicated, most of the success is directly attributable to the fidelity with which atmospheric motion is represented by the Euler equations of compressible fluid motion. The only filtering approximation it is found convenient to introduce is the hydrostatic assumption which removes all but the horizontally propagating sound wave and causes negligible error at scales currently resolvable in global models: the resulting equations are known as the primitive equation set.

Because of the diversity of solutions supported by the primitive equations, much of our theoretical understanding has been derived from solutions obtained after judicious simplification of the equations. For instance, by assuming motion on a spherical planet to be horizontally non-divergent and the air density uniform, the dynamics of planetary Rossby waves were exposed (Rossby 1939; Haurwitz 1940). A less severe filtering approximation, based on a scale analysis, was put forward by Charney (1948) and this led to the modern quasi-geostrophic theory (see also Charney & Stern 1962; White 1977) which provides the basic mathematical framework for much of our understanding of large-scale atmospheric motion (e.g. baroclinic instability theory).

Phillips (1963) identified two types of quasi-geostrophic equations; Type I being the above set due to Charney and Type II whose properties were elucidated by Burger (1958). Both types have analogues of the global energy conservation and Lagrangian potential vorticity conservation properties of the primitive equations yet neither is sufficiently accurate as a model of the entire terrestrial atmosphere or ocean. Type I is not strictly valid over a wide range of latitudes; it neglects the full variation of the Coriolis parameter except where differentiated, and linearizes the vertical advection of entropy about an assumed basic-state entropy field which is dependent on height alone. In particular, the Coriolis parameter is constant in the geostrophic wind relation unlike the Type II equations. The principal difficulty with the Type II equations is the severity of the approximation to the momentum equations – all the acceleration terms are omitted and the horizontal wind is assumed to be geostrophic in the continuity of mass and thermodynamic equations.

A less restrictive class of geostrophically balanced models which could be used with more confidence in the spherical domain was identified by Lorenz (1960). These conserve global energy though they do not have an analogue of potential vorticity conservation on fluid parcels. Numerous other forms of balanced model have been proposed and have been classified by McWilliams & Gent (1980) and reviewed by Gent & McWilliams (1983). In this paper attention will be focused on the semi-geostrophic equations proposed by Eliassen (1948) and developed through the use of the geostrophic momentum coordinate transformation by Hoskins (1975). The importance of the semi-geostrophic equations stems from their conceptual simplicity, and analytic solutions have provided much insight into frontogenesis (Hoskins & Bretherton 1972) and flow over two-dimensional orography (Pierrehumbert 1985). It is now known that the semi-geostrophic equations admit discontinuous solutions which 'in the real world' correspond to atmospheric fronts and inversions. An extended semi-geostrophic theory was developed by Cullen & Purser (1984) and has led to a geometrical technique for solving the Lagrangian form of the semi-geostrophic equations (Cullen, Chynoweth & Purser 1987*a*; Chynoweth 1987; Shutts, Cullen & Chynoweth 1988). Cullen & Purser recognized that at any instant, a fluid parcel could be characterized by the vector gradient of a modified pressure function (P). The positions of all fluid parcels within a convex region of space were then proved to be uniquely determined by the requirement that P be convex within that domain. The convexity of P corresponds to the physical necessity that the fluid parcels be arranged into a convectively and symmetrically stable state. Shutts & Cullen (1987) and Cullen *et al.* (1987*b*) show this to be a minimum energy state.

The geometrical solution technique allows the implicit geostrophic adjustment inherent in the conventional semi-geostrophic theory to extend to convectively unstable situations: model elements are able to perform a type of penetrative convection. Artificial viscosity is not required and discontinuous solutions are readily described.

Salmon (1983, 1985) has shown how semi-geostrophic theory may be extended to situations where the Coriolis parameter is a function of horizontal position. Starting with a Lagrangian formulation of Hamilton's principle for the Euler equations, he introduces approximations which preserve the symmetries of the Hamiltonian so that analogues of the conservation laws are obtained. A new set of balanced equations is obtained (Salmon 1985) which take on a simple form in suitably chosen transformed space coordinates.

In this paper, Hamilton's principle is used to formulate a new set of balanced equations for planetary flow which have the usual f-plane semi-geostrophic equations as a special case. The procedure adopted is slightly different from Salmon's: a canonical form for the principle is sought *before* making any approximations to the Hamiltonian. This has the effect of making explicit the Lagrangian nature of the basic assumptions underlying the validity of semi-geostrophic theory. The geostrophic approximation does not have to be introduced directly into the Lagrangian of Hamilton's principle. It is, in fact, furnished by the variational principle itself. Also there is an explicit recognition that the planetary rotation and gravitation vectors are not collinear.

The equations derived here are different from those of Salmon and have more in common with the f-plane semi-geostrophic theory.

2. New balanced equations derived from Hamilton's principle

2.1. *Primitive equations*

Following Salmon (1983), the extended form of Hamilton's principle will be used for which particle positions and conjugate momenta are independent coordinates. For a compressible, rotating fluid under the action of a gravitational field, Hamilton's principle may be written as

$$\delta \int_{t_0}^{t_1} \mathrm{d}\tau \left\{ \int_D \left[(u-\Omega y)\frac{\partial x}{\partial \tau} + (v+\Omega x)\frac{\partial y}{\partial \tau} + w\frac{\partial z}{\partial \tau}\right] \mathrm{d}\Gamma - H(\tau) \right\} = 0, \tag{1}$$

where the Hamiltonian function H is given by

$$H(\tau) = \int_D \mathrm{d}\Gamma \left[\tfrac{1}{2}(u^2+v^2+w^2) + U + \Phi - p\left(J\left(\frac{x,y,z}{a,b,c}\right) - \alpha\right) - T(S-S_0)\right] \tag{2}$$

using a Cartesian representation (x,y,z) for the physical space coordinates of a fluid particle in a system with rotation vector $(0,0,\Omega)$ and where (u,v,w) is the velocity of the fluid, $U(\alpha,S)$ is the internal energy, α is the specific volume, S is the gas entropy, $\Phi(x,y,z)$ is the gravitational potential (with centrifugal term absorbed) and J is the Jacobian of the transformation between (x,y,z) and particle label space (a,b,c). Unless otherwise indicated, all dependent variables in (1) and (2) are to be regarded as functions of particle label and time τ. Therefore, we may tentatively identify (u,v,w) with

$$\left(\frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}, \frac{\partial z}{\partial \tau}\right).$$

Integration over the domain (denoted by D) is with respect to particle label coordinate and $\mathrm{d}\Gamma = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \mathrm{d}(\text{mass})$ by definition.

Continuity of mass and conservation of entropy, represented by

$$J\left(\frac{x, y, z}{a, b, c}\right) = \alpha \tag{3}$$

and

$$S = S_0(a, b, c) \tag{4}$$

respectively, are enforced through the Lagrange multipliers $p(a, b, c, \tau)$ and $T(a, b, c, \tau)$ appearing in the Hamiltonian. Variations made in (1) are such that u, v, w, x, y, z, α and S are treated as independent.

Consider, for instance, variations with respect to x; (1) then gives

$$\int_{t_0}^{t_1} \mathrm{d}\tau \left\{ \int_D \left[\delta x \left(-\frac{\partial(u-\Omega y)}{\partial \tau} + \Omega \frac{\partial y}{\partial \tau} - \frac{\partial \Phi}{\partial x} \right) + p\delta \left\{ J\left(\frac{x, y, z}{a, b, c}\right) \right\} \right] \mathrm{d}\Gamma \right\} = 0 \tag{5}$$

and since

$$p\delta J\left(\frac{x, y, z}{a, b, c}\right) = J\left(\frac{p\delta x, y, z}{a, b, c}\right) - \delta x J\left(\frac{p, y, z}{a, b, c}\right)$$

or, using (3) and the rule for multiplication of Jacobians,

$$p\delta J\left(\frac{x, y, z}{a, b, c}\right) = J\left(\frac{p\delta x, y, z}{a, b, c}\right) - \delta x \alpha \frac{\partial p}{\partial x}. \tag{6}$$

If δx is required to vanish on the boundary of D then the Euler–Lagrange equation derived from (5) is

$$\frac{\partial u}{\partial \tau} - 2\Omega \frac{\partial y}{\partial \tau} + \frac{\partial \Phi}{\partial x} + \alpha \frac{\partial p}{\partial x} = 0. \tag{7}$$

Similarly, independent variations of y and z give

$$\frac{\partial v}{\partial \tau} + 2\Omega \frac{\partial x}{\partial \tau} + \frac{\partial \Phi}{\partial y} + \alpha \frac{\partial p}{\partial y} = 0 \tag{8}$$

and

$$\frac{\partial w}{\partial \tau} + \frac{\partial \Phi}{\partial z} + \alpha \frac{\partial p}{\partial z} = 0. \tag{9}$$

Furthermore, independent variations with respect to the remaining functions u, v, w, α and S give

$$\delta u: \quad \frac{\partial x}{\partial \tau} = u, \tag{10}$$

$$\delta v: \quad \frac{\partial y}{\partial \tau} = v, \tag{11}$$

$$\delta w: \quad \frac{\partial z}{\partial \tau} = w, \tag{12}$$

$$\delta \alpha: \quad p = -\frac{\partial U}{\partial \alpha}, \tag{13}$$

and

$$\delta S: \quad T = \frac{\partial U}{\partial S}. \tag{14}$$

Equations (10)–(12) provide no new information but merely demonstrate self-consistency of the definitions; (13) and (14) show that p and T are simply the pressure and temperature – consistent with their thermodynamic definition. Therefore, (7)–(9) may be written vectorially, with the aid of (10)–(14), as

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{V}+2\boldsymbol{\Omega}\wedge\boldsymbol{V}+\nabla\Phi+\alpha\nabla p=0, \tag{15}$$

where

$$\frac{\partial}{\partial\tau}\equiv\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+\boldsymbol{V}\cdot\nabla,\quad t=\tau,$$

$$\boldsymbol{V}=(u,v,w),\quad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right),\quad \boldsymbol{\Omega}=(0,0,\Omega)$$

and all dependent variables are regarded as functions of (x,y,z,t). Equation (15) is the Euler equation of motion for an inviscid, rotating fluid.

The meteorological primitive equations can be obtained for the f-plane case (where $\nabla\Phi \parallel \boldsymbol{\Omega}$) by defining $\Phi=gz$ and omitting all terms in w from (1). This removes the geopotential terms from (7) and (8) and reduces (9) to the hydrostatic equation:

$$g+\alpha\frac{\partial p}{\partial z}=0. \tag{16}$$

Extension to planetary flows where the gravitation and rotation vectors are not collinear is achieved by expressing (1) in spherical polar coordinates; omitting terms in $\mathrm{D}r/\mathrm{D}t$ (the local vertical velocity); expanding the radial coordinate r about a mean planetary radius a and neglecting terms of the order of (scale height of the atmosphere)/a.

The hydrostatic approximation is valid, therefore, if the kinetic energy is well approximated by that associated with the horizontal components of the flow. If h is a typical depth scale and l a typical horizontal lengthscale of an atmospheric motion system, then conventional scale analysis requires that $h^2/l^2 \ll 1$ for the validity of the hydrostatic assumption (Holton 1979). Whilst this inequality correctly identifies hydrostatic motion in the terrestrial atmosphere, it is not a necessary condition as was pointed out by Phillips (1963). The Hamiltonian formulation of the primitive equations emphasizes the dependence of the hydrostatic assumption on the smallness of the kinetic energy in the vertical motion.

As shown by Salmon (1983, 1985, 1988), approximations that preserve the time and particle labelling symmetries of the Lagrangian in Hamilton's principle automatically imply that the Hamiltonian is an integral invariant and that an analogue of potential vorticity conservation exists. These properties will be considered later for the new equations to be derived. Before deriving these new equations it is instructive to apply the same methods and coordinate transformations to be used to a derivation of the Phillips Type II quasi-geostrophic equations. A simple version of these equations, with slightly greater formal accuracy, results.

2.2. *Phillips Type II quasi-geostrophic equations*

The 'potential' energy term $U+\Phi$ in (2) can be partitioned into a basic state component (for which the gas entropy is a function of pressure alone) and an available potential energy component which can be converted into kinetic energy (Lorenz 1955). For some scales of motion the available potential energy may dominate the kinetic energy; this happens when the Burger number B_u, given by

$$B_u=\left(\frac{\bar{N}h}{2\Omega l}\right)^2,$$

satisfies $B_u \ll 1$ where $\bar{N}$ is a mean buoyancy frequency (Burger 1958). In the atmosphere this assumption is only valid for the largest scales of planetary Rossby wave motion; on the other hand extensive regions of the ocean have small Burger number.

Motivated by the smallness of the kinetic energy in these situations, all terms in (1) involving u, v or w may be removed leaving

$$\delta \int_{t_0}^{t_1} \mathrm{d}\tau \left\{ \int_D \Omega \left(x\frac{\partial y}{\partial \tau} - y\frac{\partial x}{\partial \tau} \right) \mathrm{d}\Gamma - H(\tau) \right\} = 0, \tag{17}$$

where
$$H(\tau) = \int_D \mathrm{d}\Gamma \left[U + \Phi - p\left\{ J\left(\frac{x, y, z}{a, b, c}\right) - \alpha \right\} - T(S - S_0) \right]. \tag{18}$$

The Lagrangian in (17) is closely related to that in equation (4.1) of Salmon (1983) which was for a shallow-water fluid system with a spatially varying Coriolis parameter and rigid boundary at $z = 0$. Independent variations with respect to x, y and z give

$$\delta x: \quad 2\Omega\frac{\partial y}{\partial \tau} - \frac{\partial \Phi}{\partial x} - \alpha\frac{\partial p}{\partial x} = 0, \tag{19}$$

$$\delta y: \quad -2\Omega\frac{\partial x}{\partial \tau} - \frac{\partial \Phi}{\partial y} - \alpha\frac{\partial p}{\partial y} = 0, \tag{20}$$

$$\delta z: \quad -\frac{\partial \Phi}{\partial z} - \alpha\frac{\partial p}{\partial z} = 0. \tag{21}$$

Equations (19)–(21) may be combined to give

$$2\boldsymbol{\Omega} \wedge \boldsymbol{V} + \nabla\Phi + \alpha\nabla p = 0 \tag{22}$$

which is a vector statement of geostrophic and hydrostatic balance. The Eulerian forms of the continuity and thermodynamic equations are

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\alpha = \alpha\nabla\cdot\boldsymbol{V} \tag{23}$$

and
$$\left(\frac{\partial}{\partial t} + \nabla\cdot\boldsymbol{V}\right)S = 0, \tag{24}$$

which, together with (22), the definition of S and the perfect gas equation, form a closed set of equations. Apart from the inclusion of a small Coriolis term associated with the horizontal component of $\boldsymbol{\Omega}$, the spherical polar expression of this set is widely known as the Phillips Type II quasi-geostrophic equations or, the Burger equations. It is convenient to express them in spherical polar coordinates because Φ is radially symmetric and atmospheres are highly stratified in the radial direction. A consequence of this stratification is that $\boldsymbol{V}$ is dominated by its horizontal components and a unique geostrophic wind $\boldsymbol{V}_g$ can be defined such that

$$2(\boldsymbol{k}\cdot\boldsymbol{\Omega})\, \boldsymbol{V}_g = \alpha\boldsymbol{k} \wedge \nabla p, \tag{25}$$

where $\boldsymbol{k}$ is the local unit vector in the direction of $\nabla\Phi$ (Phillips 1963) and $\boldsymbol{V}_g$ may be used in (23) and (24) in place of $\boldsymbol{V}$. (Note that (22) does not, by itself, define $\boldsymbol{V}$ uniquely.)

An alternative and illuminating approach is to express (22)–(24) in (X, Y, Z) coordinates, where

$$x = X, \quad y = Y, \quad Z = Z_a[1-(p/p_*)^{(\gamma-1)/\gamma}], \tag{26}$$

where γ is the ratio of specific heats (C_p/C_v), p_* is a constant reference pressure taken here to be mean sea-level pressure, Z_a is given by

$$Z_a = \frac{\gamma p_* \alpha_*}{(\gamma-1)\,g},$$

g is the acceleration due to gravity and α_* is the mean specific volume at p_*. The partial derivatives can then be shown to transform according to

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} + \Pi(p)\frac{\partial p}{\partial x}\frac{\partial}{\partial Z} \tag{27}$$

(similarly for y), and

$$\frac{\partial}{\partial z} = \Pi(p)\frac{\partial p}{\partial z}\frac{\partial}{\partial Z}, \tag{28}$$

where

$$\Pi(p) = Z_a\frac{(1-\gamma)}{\gamma p_*}\left(\frac{p}{p_*}\right)^{-1/\gamma}$$

Using (21), (27) and (28) we obtain

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial X} + \Pi(p)\frac{\partial p}{\partial x}\frac{\partial \Phi}{\partial Z}, \tag{29}$$

$$\frac{\partial \Phi}{\partial z} = \Pi(p)\frac{\partial p}{\partial z}\frac{\partial \Phi}{\partial Z} = -\alpha\frac{\partial p}{\partial z}; \tag{30}$$

(30) implies that

$$\Pi(p)\frac{\partial \Phi}{\partial Z} = -\alpha \tag{31}$$

which, on substitution into (29) gives

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \Phi}{\partial x} + \alpha\frac{\partial p}{\partial x} \tag{32}$$

and similarly for the y-component.

Equations (19) and (20) may now be simplified to

$$2\Omega v = \frac{\partial \Phi}{\partial X} \tag{33}$$

and

$$2\Omega u = -\frac{\partial \Phi}{\partial Y}, \tag{34}$$

and using the perfect gas equation and the definition of potential temperature θ, (31) may be rewritten as

$$\frac{\partial \Phi}{\partial Z} = \frac{g\theta}{\theta_0}, \tag{35}$$

where $\theta_0 = p_*\alpha_*/R$ and R is the gas constant. Hoskins & Bretherton (1972) introduced the pressure-dependent local vertical coordinate Z because it combines some of the advantages of pressure coordinates (anelastic continuity equation, simplified pressure gradient force terms) with its tendency to approximate physical height in the troposphere and give a hydrostatic relation involving potential temperature. Cartesian f-plane problems solved using this transformation invariably have horizontal z-surfaces which are quasi-parallel to Z-surfaces so that horizontal differentiation holding Z constant can be visualized as virtually the same as holding z constant. In its application to planetary flows here, z surfaces are *not* horizontal and $\partial/\partial x$ is vastly different from $\partial/\partial X$ (figure 1 *a*) due to the near-sphericity of Z-surfaces. Similarly, one should be aware of the difference between $\partial/\partial Z$ holding X and Y constant and the usual vertical derivative in Z-coordinates. We may refer to $\partial/\partial Z$ as an axial derivative to distinguish it from differentiation with respect to Z in the direction of $\nabla\Phi$ (the local vertical, see figure 1 *b*). It is easy to show from figure 1 (*b*) that

$$\frac{\partial}{\partial Z}=\left(\frac{\partial}{\partial Z}\right)_{\mathrm{LV}}+\frac{\cot\phi}{r}\frac{\partial}{\partial\phi}, \tag{36}$$

where ϕ is latitude, r is the distance from the Earth's centre and LV denotes differentiation in the direction of the local vertical. This implies (using (35)) a hydrostatic relation of the form

$$-\frac{g\theta}{\theta_0}+\left(\frac{\partial\Phi}{\partial Z}\right)_{\mathrm{LV}}=-\frac{\cot\phi}{r}\frac{\partial\Phi}{\partial\phi}=-2\Omega\cos\phi u_{\mathrm{g}}, \tag{37}$$

where u_{g} is the conventional zonal geostrophic wind speed. It is noteworthy that the axial hydrostatic balance, (35), implies a radial balance which includes the small Coriolis term due to zonal motion.

Consider now the continuity equation (23) in a Cartesian space whose coordinates are (X, Y, Z). This is simply

$$\frac{\partial\rho'}{\partial T}+\frac{\partial(\rho' u)}{\partial X}+\frac{\partial(\rho' v)}{\partial Y}+\frac{\partial(\rho' w)}{\partial Z}=0, \tag{38}$$

where $$w=\frac{\mathrm{D}Z}{\mathrm{D}T}\quad\left(\frac{\mathrm{D}}{\mathrm{D}T}\equiv\frac{\partial}{\partial\tau}\right)$$

and ρ' is the density in (X, Y, Z)-space given by

$$\rho'=\alpha^{-1}J\left(\frac{x,y,z}{X,Y,Z}\right).$$

But $$J\left(\frac{x,y,z}{X,Y,Z}\right)=\frac{\partial z}{\partial Z}=\left(\Pi(p)\frac{\partial p}{\partial z}\right)^{-1}$$

and therefore

$$\rho'=\left(\Pi(p)\,\alpha\frac{\partial p}{\partial z}\right)^{-1}=\rho(Z)\left(\frac{1}{g}\frac{\partial\Phi}{\partial z}\right)^{-1}, \tag{39}$$

where $\rho(Z)$ is the pseudo-density defined in Hoskins & Bretherton (1972) as

$$\rho(Z)=\alpha_*^{-1}(p/p_*)^{1/\gamma}. \tag{40}$$

Now $\partial\Phi/\partial z=g\sin\phi[1+O(h/a)]$, where ϕ is latitude and a is the mean radius of the Earth, and so

$$\rho'\approx\frac{\rho(Z)}{\sin\phi}. \tag{41}$$

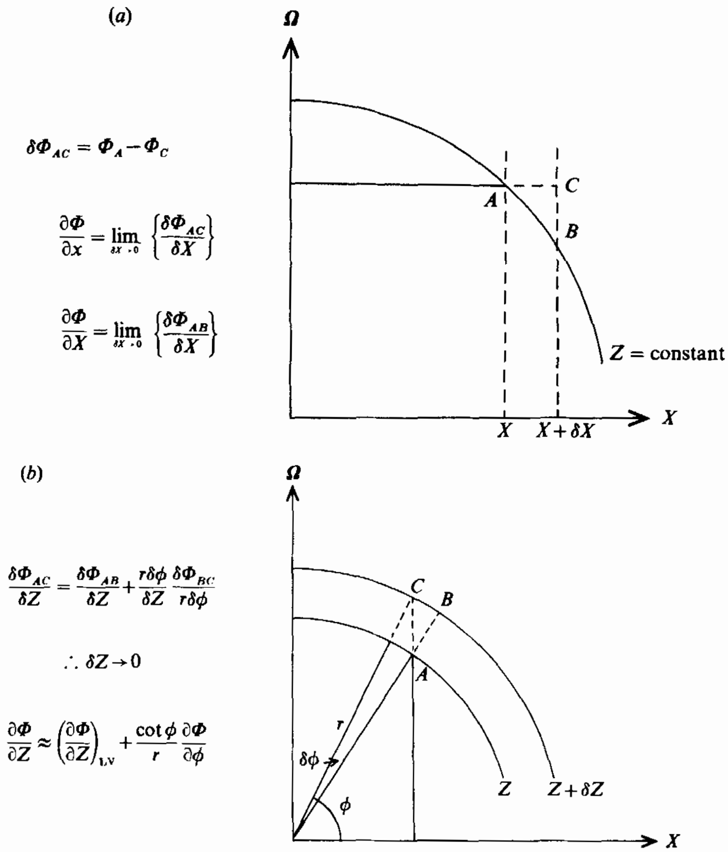

FIGURE 1. (*a*) Schematic diagram showing the difference between differentiation with respect to x holding z or Z constant. (*b*) Schematic diagram showing the difference between differentiation with respect to Z in the vertical and axial directions.

Also $\sin\phi = \{1-(X^2+Y^2)/a^2+O(h/a)\}^{\frac{1}{2}}$ and since $h/a \sim 10^{-3}$, $\sin\phi$ may be regarded as a function of X and Y, so that $\rho' = \rho'(X,Y,Z)$. The continuity equation in transformed coordinates, (38), then assumes the anelastic form

$$\frac{\partial(\rho' u)}{\partial X}+\frac{\partial(\rho' v)}{\partial Y}+\frac{\partial(\rho' w)}{\partial Z} = 0, \tag{42}$$

which, on substitution for u and v from (33) and (34), becomes

$$\frac{1}{2\Omega}J\left(\frac{\rho', \Phi}{X, Y}\right) = \frac{\partial(\rho' w)}{\partial Z}. \tag{43}$$

To compare this equation with that derived by Phillips (1963), a transformation of X and Y to latitude ϕ and longitude λ is required. The mapping

$$X = a\cos\lambda\cos\phi + O(h/a), \quad Y = a\sin\lambda\cos\phi + O(h/a)$$

has a Jacobian of transformation given by

$$J\left(\frac{X, Y}{\lambda, \phi}\right) = a^2\cos\phi\sin\phi + O(h/a). \tag{44}$$

Multiplying (43) by this Jacobian and simplifying gives

$$J\left(\frac{\rho', \Phi}{\lambda, \phi}\right) = -\frac{\partial \Phi}{\partial \lambda}\frac{\partial}{\partial \phi}\left[\frac{\rho(Z)}{\sin\phi}\right] = 2\Omega a^2 \cos\phi \sin\phi \frac{\partial(\rho' w)}{\partial Z}, \tag{45}$$

which, on using (36) and (41), gives

$$\frac{1}{\sin^2\phi}\frac{\partial \Phi}{\partial \lambda} = 2\Omega a^2\left[\left\{\frac{1}{\rho}\frac{\partial(\rho w)}{\partial Z}\right\}_{\mathrm{LV}} + \frac{\cot\phi}{a}\frac{\partial w}{\partial \phi} - \frac{w}{a}\cot^2\phi\right]. \tag{46}$$

This equation is the compressible equivalent of equation (6.3) on p. 163 of Phillips' review except for the second and third terms on the right-hand side. As alluded to earlier, these result from the inclusion of a small Coriolis term. To show, this, (46) may be derived by substituting the following expressions for u_g and v_g:

$$2\Omega \sin\phi u_g = -\frac{1}{a}\frac{\partial \Phi}{\partial \phi}, \tag{47}$$

$$2\Omega \sin\phi v_g = \frac{1}{a\cos\phi}\frac{\partial \Phi}{\partial \lambda} + 2\Omega\cos\phi w, \tag{48}$$

into the continuity equation:

$$\frac{2w}{a} + \frac{1}{\rho}\left\{\frac{\partial(\rho w)}{\partial Z}\right\}_{\mathrm{LV}} + \frac{1}{a\cos\phi}\left\{\frac{\partial(v_g\cos\phi)}{\partial\phi} + \frac{\partial u_g}{\partial \lambda}\right\} = 0, \tag{49}$$

noting the inclusion of the tiny metric term $2w/a$ for air parcels moving radially.

Since $S = C_p \ln\theta$, the thermodynamic equation (24) may be written as

$$\frac{\partial \theta}{\partial T} + \frac{1}{2\Omega}J\left(\frac{\Phi, \theta}{X, Y}\right) + w\frac{\partial \theta}{\partial Z} = 0$$

or, using (44) and (35)

$$\frac{\partial^2 \Phi}{\partial T \partial Z} + \frac{1}{2\Omega a^2 \sin\phi\cos\phi}J\left(\frac{\Phi, (\partial\Phi/\partial Z)}{\lambda, \phi}\right) + w\frac{\partial^2\Phi}{\partial Z^2} = 0. \tag{50}$$

Equations (45) and (50) form a closed set involving Φ and w.

Although, as we have seen, this is a very severely approximated balanced set, it is widely used in oceanography (e.g. Pedlosky 1979; Anderson & Killworth 1979) and has, on occasion, been used to study ultra-long waves in the atmosphere (e.g. Bates 1977; Lynch 1979).

Global energy and Lagrangian potential vorticity conservation follow from time and particle label symmetries. The details are not reproduced here though the method is essentially the same as in §2.4.

2.3. *New 'planetary semi-geostrophic' equations*

Without any regard for accuracy at this stage, the first approximation to be made is to omit all terms in w from (1). As was seen in §2.1, this gives the hydrostatic approximation for an f-plane system whose axis of rotation is in the z-direction. Hamilton's principle then requires that

$$\delta\int_{t_0}^{t_1} \mathrm{d}\tau\left\{\int_D\left[(u-\Omega y)\frac{\partial x}{\partial \tau} + (v+\Omega x)\frac{\partial y}{\partial \tau}\right]\mathrm{d}\Gamma - H(\tau)\right\} = 0, \tag{51}$$

where now

$$H(\tau) = \int_D \mathrm{d}\Gamma\left[\tfrac{1}{2}(u^2+v^2)+U+\Phi-p\left(J\left(\frac{x,y,z}{a,b,c}\right)-\alpha\right)-T(S-S_0)\right]. \quad (52)$$

We now seek a transformation to canonical form by letting

$$(u-\Omega y)\frac{\partial x}{\partial \tau}+(v+\Omega x)\frac{\partial y}{\partial \tau} = \frac{1}{2\omega}\left(M\frac{\partial N}{\partial \tau}-N\frac{\partial M}{\partial \tau}\right)+\frac{\partial B}{\partial \tau}+C, \quad (53)$$

where M and N are to be the new canonical coordinates, ω is an unknown constant, B is an arbitrary function of the old and new coordinates and C represents the residual terms which, when neglected, define the approximation.

Inspired by the canonical form of (17) which leads to the Burger equations, we choose

$$M = \omega x+F(a,b,c,\tau), \quad N = \omega y+G(a,b,c,\tau), \quad (54)$$

where F and G are unknown functions to be determined. Substitution of (54) into (53) suggests the choice

$$\omega = 2\Omega, \quad G = -u, \quad F = v$$

which, on rearrangement, gives

$$(u-\Omega y)\frac{\partial x}{\partial \tau}+(v+\Omega x)\frac{\partial y}{\partial \tau} = \frac{1}{4\Omega}\left(M\frac{\partial N}{\partial \tau}-N\frac{\partial M}{\partial \tau}\right)+\frac{\partial}{\partial \tau}\frac{(xu+yv)}{2}-\frac{1}{4\Omega}\left(u\frac{\partial v}{\partial \tau}-v\frac{\partial u}{\partial \tau}\right), \quad (55)$$

with $M = 2\Omega x+v$ and $N = 2\Omega y-u$.

But the last term in (55) may be written as

$$\frac{u^2+v^2}{4\Omega}\frac{\partial \chi}{\partial \tau}$$

where χ is the angle (measured anticlockwise) that the velocity vector, projected onto the (x,y)-plane, makes with an arbitrary fixed direction. Since in Hamilton's principle, variations are taken to vanish at the endpoint times t_0 and t_1, the term $(\partial/\partial\tau)(xu+yv)/2$ integrates out and (51) and (52) become

$$\delta\int_{t_0}^{t_1}\mathrm{d}\tau\left\{\frac{1}{4\Omega}\int\left[M\frac{\partial N}{\partial \tau}-N\frac{\partial M}{\partial \tau}\right]\mathrm{d}\Gamma-H'(\tau)\right\} = 0, \quad (56)$$

$$H'(\tau) = \int_D\left\{\tfrac{1}{2}\{(2\Omega y-N)^2+(M-2\Omega x)^2\}\left(1+\frac{1}{2\Omega}\frac{\partial \chi}{\partial \tau}\right)\right.$$
$$\left.+U+\Phi-p\left(J\left(\frac{x,y,z}{a,b,c}\right)-\alpha\right)-T(S-S_0)\right\}\mathrm{d}\Gamma. \quad (57)$$

Our second approximation requires that the term $(2\Omega)^{-1}\partial\chi/\partial\tau$ be neglected under the assumption that

$$\frac{1}{2\Omega}\left|\frac{\partial \chi}{\partial \tau}\right| \ll 1, \quad (58)$$

condition (A).

Physically, this implies that the rate of turning of the wind vector following fluid particles (in the (x,y)-plane) is small compared with twice the angular rotation rate of the system. Alternatively, it requires that the centrifugal force of the relative motion be small compared with the Coriolis force.

Independent variations of M, N, x, y and z are easily seen to give

$$\delta M: \quad \frac{\partial N}{\partial \tau} = 2\Omega(M-2\Omega x), \tag{59}$$

$$\delta N: \quad \frac{\partial M}{\partial \tau} = 2\Omega(2\Omega y-N), \tag{60}$$

$$\delta x: \quad 2\Omega v = 2\Omega(M-2\Omega x) = \frac{\partial \Phi}{\partial x}+\alpha\frac{\partial p}{\partial x}, \tag{61}$$

$$\delta y: \quad -2\Omega u = -2\Omega(2\Omega y-N) = \frac{\partial \Phi}{\partial y}+\alpha\frac{\partial p}{\partial y}, \tag{62}$$

$$\delta z: \quad \frac{\partial \Phi}{\partial z}+\alpha\frac{\partial p}{\partial z} = 0. \tag{63}$$

For condition (A) to be consistent 'after the fact' we require, using (61) and (62), that

$$\frac{1}{2\Omega}\left|\frac{\partial \hat{\nu}}{\partial \tau}\right| \ll 1, \tag{64}$$

condition (B), where $\hat{\nu}$ is the angle (measured anticlockwise) that the vector $\nabla\Phi+\alpha\nabla p$ makes with any fixed direction when projected into the equatorial plane. Conditions (A) and (B) will be treated as independent since (A) is required to obtain the Euler equations (59)–(63) in the first instance and (B) is implied on substitution for u and v from (61) and (62).

Introducing the coordinate transformation described in §2.2 to simplify (61)–(63) gives

$$2\Omega(M-2\Omega X) = \frac{\partial \Phi}{\partial X}, \tag{65}$$

$$2\Omega(2\Omega Y-N) = \frac{\partial \Phi}{\partial Y}, \tag{66}$$

$$\frac{g\theta}{\theta_0} = \frac{\partial \Phi}{\partial Z}, \tag{67}$$

and the evolution equations (59) and (60) may be written as

$$\frac{\mathrm{D}N}{\mathrm{D}t} = \frac{\partial \Phi}{\partial X}, \tag{68}$$

$$\frac{\mathrm{D}M}{\mathrm{D}t} = -\frac{\partial \Phi}{\partial Y}, \tag{69}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative.

Equations (65)–(69), together with the continuity of mass and thermodynamic equations, are identical to the f-plane semi-geostrophic equations of Hoskins (1975). In that case, the velocity component whose contribution to the kinetic energy in the Hamiltonian is neglected is directed along the local vertical. This enforces the conventional hydrostatic assumption: the accuracy of the semi-geostrophic equations then depends primarily on the smallness of particle accelerations with respect to the Coriolis force.

Condition (58) is one of two conditions Hoskins identifies as being necessary for the validity of the semi-geostrophic equations. His second condition requires that:

$$|\mathrm{D}V/\mathrm{D}t| \ll fV$$

where $V = |\boldsymbol{V}|$ and can be shown to be implied by conditions (58) and (64).

Planetary-scale flows are also governed by (65)–(69) but the absence of the velocity component parallel to the axis of rotation from the kinetic energy renders the system relatively less accurate than the f-plane semi-geostrophic equations. Under these circumstances the system (65)–(69) will be called the planetary semi-geostrophic (PSG) equations. Only planetary motions that are zonally elongated or have small Burger number will be accurately treated. An attempt to quantify the distortion inherent in these planetary semi-geostrophic equations with respect to the primitive equations is described in §3.

At this point it is worth emphasizing the extreme simplicity of the Lagrangian form of the PSG equations. This, together with the conservation properties to be demonstrated, greatly compensates for the loss of accuracy of this system of equations. Accuracy alone should not be considered the sole aim when devising balanced equation sets since the primitive equations can be readily integrated on the computer – in spite of the limitations imposed by fast-moving gravity modes.

Although the PSG equations are extremely simple in the Cartesian form (65)–(69), they are – for some practical purposes – better expressed in spherical polar coordinates. As in §2.2, let

$$X = a\cos\lambda\cos\phi, \quad Y = a\sin\lambda\cos\phi$$

so that

$$\frac{\partial\Phi}{\partial X} = -\frac{1}{a}\frac{\sin\lambda}{\cos\phi}\frac{\partial\Phi}{\partial\lambda} - \frac{1}{a}\frac{\cos\lambda}{\sin\phi}\frac{\partial\Phi}{\partial\phi}, \tag{70}$$

$$\frac{\partial\Phi}{\partial Y} = \frac{\cos\lambda}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda} - \frac{1}{a}\frac{\sin\lambda}{\sin\phi}\frac{\partial\Phi}{\partial\phi} \tag{71}$$

(NB $\partial/\partial\lambda$ and $\partial/\partial\phi$ are at constant Z) and

$$\dot{X} = -a\sin\lambda\cos\phi\dot{\lambda} - a\sin\phi\cos\lambda\dot{\phi}, \tag{72}$$

$$\dot{Y} = a\cos\lambda\cos\phi\dot{\lambda} - a\sin\lambda\sin\phi\dot{\phi}, \tag{73}$$

where λ and ϕ may be identified with longitude and latitude respectively to a high degree of accuracy.

Now (68) and (69) may (using (65) and (66)) be written as

$$\frac{1}{2\Omega}\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\Phi}{\partial X} + 2\Omega\dot{X} = -\frac{\partial\Phi}{\partial Y} \tag{74}$$

and

$$\frac{1}{2\Omega}\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\Phi}{\partial Y} + 2\Omega\dot{Y} = \frac{\partial\Phi}{\partial X}. \tag{75}$$

Multiplying (74) by $\sin\lambda$ and subtracting from this (75) $\times\cos\lambda$ gives

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{-\Phi_\lambda}{2\Omega a\cos\phi}\right] + \frac{\dot{\lambda}\Phi_\phi}{2\Omega a\sin\phi} - 2\Omega a\cos\phi\dot{\lambda} = \frac{\Phi_\phi}{a\sin\phi} \tag{76}$$

on using (70)–(73). If u_g and v_g are the usual eastward and northward components of the geostrophic wind given by

$$v_g = \frac{\Phi_\lambda}{2\Omega a \sin\phi \cos\phi} \quad \text{and} \quad u_g = -\frac{\Phi_\phi}{2\Omega a \sin\phi}$$

then (76) may be written as

$$\sin\phi \frac{\mathrm{D}(v_g \sin\phi)}{\mathrm{D}t} + \frac{u_g u' \tan\phi}{a} + 2\Omega u' \sin\phi = 2\Omega u_g \sin\phi, \tag{77}$$

where the prime on u indicates the full eastward velocity component. Similarly, it can be readily shown that the zonal component of the momentum equation becomes

$$\frac{\mathrm{D}u_g}{\mathrm{D}t} - \frac{u' v_g \tan\phi}{a} - 2\Omega \sin\phi v' = -2\Omega \sin\phi v_g. \tag{78}$$

Neglecting the Coriolis term (as it appears in (37)), the vertical momentum equation (67) then expresses the conventional hydrostatic balance

$$\frac{g\theta}{\theta_0} = \left(\frac{\partial\Phi}{\partial Z}\right)_{\mathrm{LV}} \tag{79}$$

To arrive at (77) and (78) directly from the primitive equations of motion, not only does one have to introduce the geostrophic momentum assumption but also include $\sin\phi$ factors in the material derivative of the meridional momentum. This is consistent with the neglect of a contribution $v_g \cos\phi$ from the kinetic energy term in the Hamiltonian (i.e. the balanced kinetic energy for the equation set is $\frac{1}{2}[u_g^2 + (v_g \sin\phi)^2]$).

2.4. *Time and particle label symmetry*

The absence of terms in the integrand of (56) with an explicit time dependence means that the action defined by

$$\int_{t_0}^{t_1} L\left(M, N, \frac{\partial M}{\partial \tau}, \frac{\partial N}{\partial \tau}, x, y, z, \alpha, S\right) \mathrm{d}\tau,$$

where

$$L = \frac{1}{4\Omega} \int_D \left[M \frac{\partial N}{\partial \tau} - N \frac{\partial M}{\partial \tau}\right] \mathrm{d}\Gamma - H'(\tau)$$

is invariant with respect to a relabelling of the time coordinate. Consider a new time coordinate τ' related to τ by

$$\tau' = \tau + \delta\tau(\tau)$$

such that

$$\delta\tau(t_1) = \delta\tau(t_0) = 0$$

and in the limit $\delta\tau \to 0$.

Following Salmon (1983), the action difference between two identical realizations with time coordinates τ and τ' is

$$\int_{t_0}^{t_1} \mathrm{d}\tau' L\left(M, N, \frac{\partial M}{\partial \tau'}, \frac{\partial N}{\partial \tau'}, x, y, z, \alpha, S\right) - \int_{t_0}^{t_1} \mathrm{d}\tau L\left(M, N, \frac{\partial M}{\partial \tau}, \frac{\partial N}{\partial \tau}, x, y, z, \alpha, S\right)$$

$$= \int_{t_0}^{t_1} \mathrm{d}\tau \frac{\mathrm{d}\tau'}{\mathrm{d}\tau} L\left(M, N, \frac{\partial M}{\partial \tau}\frac{\mathrm{d}\tau}{\partial \tau'}, \frac{\partial N}{\partial \tau}\frac{\mathrm{d}\tau}{\partial \tau'}, x, y, z, \alpha, S\right) - \int_{t_0}^{t_1} L \,\mathrm{d}\tau$$

$$= \int_{t_0}^{t_1} \mathrm{d}\tau \left(\frac{\mathrm{d}\delta\tau}{\mathrm{d}\tau}\right)\left[L - \int_D \left\{\frac{\delta L}{\delta(\partial M/\partial\tau)}\frac{\partial M}{\partial\tau} + \frac{\delta L}{\delta(\partial N/\partial\tau)}\frac{\partial N}{\partial\tau}\right\}\mathrm{d}\Gamma\right] + O(\delta\tau^2)$$

$$= -\int_{t_0}^{t_1} \mathrm{d}\tau \left(\frac{\mathrm{d}\delta\tau}{\mathrm{d}\tau}\right) H' + O(\delta\tau^2)$$

$$= \int_{t_0}^{t_1} \mathrm{d}\tau\, \delta\tau \frac{\mathrm{d}H'}{\mathrm{d}\tau} + O(\delta\tau^2) = 0,$$

and so, since $\delta\tau$ is arbitrary,

$$\frac{\mathrm{d}H'}{\mathrm{d}\tau} = 0$$

or

$$\int_D [\tfrac{1}{2}(2\Omega y - N)^2 + \tfrac{1}{2}(M - \Omega x)^2 + U + \Phi]\,\mathrm{d}\Gamma = \text{a constant.} \tag{80}$$

Consider now a relabelling of the particles given by

$$a' = a + \delta a(a,b,c,\tau), \quad b' = b + \delta b(a,b,c,\tau), \quad c' = c + \delta c(a,b,c,\tau),$$

such that the specific volume and entropy are unaltered, i.e.

$$\delta\alpha = \delta J\left(\frac{a,b,c}{x,y,z}\right) = 0, \tag{81}$$

$$\delta S = \frac{\partial S}{\partial a}\delta a + \frac{\partial S}{\partial b}\delta b + \frac{\partial S}{\partial c}\delta c = 0, \tag{82}$$

where δ denotes the change due to relabelling. Equation (82) can be readily satisfied if c is chosen so that $S = S(c)$ thereby implying that $\delta c = 0$: the relabelling therefore involves assigning new a and b labels within S surfaces. It then follows from (81) that

$$\frac{\partial \delta a}{\partial a} + \frac{\partial \delta b}{\partial b} = 0, \tag{83}$$

which suggests the definition of a perturbation stream function $\delta\psi$ given by

$$\delta a = -\frac{\partial \delta\psi}{\partial b}, \quad \delta b = \frac{\partial \delta\psi}{\partial a}. \tag{84}$$

Time differentiation is affected by the relabelling since

$$\left.\frac{\partial M}{\partial\tau}\right|_{a,b,c} = \left.\frac{\partial M}{\partial\tau}\right|_{a',b',c'} + \frac{\partial M}{\partial a'}\frac{\partial a'}{\partial\tau} + \frac{\partial M}{\partial b'}\frac{\partial b'}{\partial\tau}$$

so that

$$\delta\left(\frac{\partial M}{\partial\tau}\right) = -\frac{\partial M}{\partial a}\frac{\partial \delta a}{\partial\tau} - \frac{\partial M}{\partial b}\frac{\partial \delta b}{\partial\tau} \tag{85}$$

and similarly for $\partial N/\partial\tau$.

The Lagrangian $L(M, N, (\partial M/\partial\tau), (\partial N/\partial\tau), x, y, z, \alpha, S)$ has no explicit dependence on a and b, implying that

$$\delta\int_{t_0}^{t_1} L\,\mathrm{d}\tau = \int_{t_0}^{t_1}\mathrm{d}\tau\left\{\int_D\left[M\delta\left(\frac{\partial N}{\partial\tau}\right) - N\delta\left(\frac{\partial M}{\partial\tau}\right)\right]\mathrm{d}\Gamma\right\},$$

which, on using (84) and (85), becomes

$$\int_{t_0}^{t_1} \mathrm{d}\tau \left\{ \int_D \left\{ -\frac{\partial \delta\psi}{\partial b}\frac{\partial}{\partial \tau}\left[M\frac{\partial N}{\partial a} - N\frac{\partial M}{\partial a}\right] + \frac{\partial \delta\psi}{\partial a}\frac{\partial}{\partial \tau}\left[M\frac{\partial N}{\partial b} - N\frac{\partial M}{\partial b}\right]\right\} \mathrm{d}\Gamma \right\}. \tag{86}$$

Integrating by parts and setting $\delta\psi = 0$ on the boundary of D leads to

$$\int_{t_0}^{t_1} \mathrm{d}\tau \left\{ \int_D \delta\psi \frac{\partial}{\partial \tau} J\left(\frac{M,N}{a,b}\right) \mathrm{d}\Gamma \right\}$$

which is zero if the evolution of the system is to be unperturbed by relabelling particles. Since $\delta\psi$ is arbitrary

$$\frac{\partial}{\partial \tau} J\left(\frac{M,N}{a,b}\right) = 0,$$

but

$$\frac{\partial S}{\partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial S}{\partial c}\right) = 0$$

and so

$$\frac{\partial}{\partial \tau}\left\{J\left(\frac{M,N}{a,b}\right)\frac{\partial S}{\partial c}\right\} = \frac{\partial}{\partial \tau} J\left(\frac{M,N,S}{a,b,c}\right) = \frac{\partial}{\partial \tau}\left\{\alpha J\left(\frac{M,N,S}{x,y,z}\right)\right\} = 0 \tag{87}$$

using the fact that S is a function of c alone. Equation (87) expresses the conservation of potential vorticity (q) following fluid parcels. q may be expressed in (X, Y, Z)-coordinates by noting that

$$q = \alpha J\left(\frac{M,N,S}{x,y,z}\right) = \alpha J\left(\frac{X,Y,Z}{x,y,z}\right) J\left(\frac{M,N,S}{X,Y,Z}\right) = \frac{1}{\rho'} J\left(\frac{M,N,S}{X,Y,Z}\right). \tag{88}$$

This may be re-expressed as the determinant of a Hessian matrix by noting that (65)–(67) may be written as

$$2\Omega M = \frac{\partial P}{\partial X}, \quad 2\Omega N = \frac{\partial P}{\partial Y}, \quad gS = \frac{\partial P}{\partial Z} \tag{89}$$

(letting $S = \theta/\theta_0$) where $P = \Phi + 2\Omega^2(X^2 + Y^2)$ so that (88) becomes

$$q = \frac{\det \boldsymbol{Q}}{4\Omega^2 \rho' g} \tag{90}$$

where

$$\boldsymbol{Q} = \begin{pmatrix} P_{XX} & P_{XY} & P_{XZ} \\ P_{YX} & P_{YY} & P_{YZ} \\ P_{ZX} & P_{ZY} & P_{ZZ} \end{pmatrix}.$$

3. Planetary semi-geostrophic eigenfunctions and their accuracy

To provide some quantitative measure of the distortion inherent in the planetary semi-geostrophic equations some standard eigenvalue problems in the spherical domain have been solved using linearized equations, and compared to eigensolutions of the corresponding primitive equation problems. Three physical problems are examined; non-divergent Rossby–Haurwitz waves with a barotropic basic state atmosphere at rest, stationary planetary Rossby waves for a uniformly stratified atmosphere in solid rotation and equatorially trapped waves under the shallow-water approximation.

3.1. *Rossby–Haurwitz waves*

Non-divergent, barotropic flow on the sphere possesses exact travelling wave solutions whose stream function takes the form of a spherical harmonic. The linearized planetary semi-geostrophic equations (in spherical polar coordinates) corresponding to this problem are

$$\sin^2\phi\frac{\partial v_g}{\partial t}+2\Omega u'\sin\phi = 2\Omega u_g\sin\phi, \tag{91}$$

$$\frac{\partial u_g}{\partial t}-2\Omega v'\sin\phi = -2\Omega\sin\phi v_g, \tag{92}$$

$$\frac{\partial u'}{\partial\lambda}+\frac{\partial(v'\cos\phi)}{\partial\phi} = 0, \tag{93}$$

where $$u_g\sin\phi = -\frac{1}{2\Omega a}\frac{\partial\Phi'}{\partial\phi},\quad v_g\sin\phi = \frac{1}{2\Omega a\cos\phi}\frac{\partial\Phi'}{\partial\lambda}.$$

Equations (91) and (92) give expressions for u' and v' (respectively) in terms of the perturbation geopotential Φ'; substitution into (93) then gives a partial differential equation in Φ'. Assume now that Φ' is a travelling wave of the form

$$\Phi' = \mathrm{Re}\,[G_m(\mu)\,\mathrm{e}^{\mathrm{i}(m\lambda-\sigma t)}],$$

where $G_m(\mu)$ is the wave amplitude and $\mu = \sin\phi$; then it can be shown that $G_m(\mu)$ satisfies the equation

$$(1-\mu^2)\frac{\mathrm{d}^2G_m}{\mathrm{d}\mu^2}-\frac{2}{\mu}\frac{\mathrm{d}G_m}{\mathrm{d}\mu}+\left(\alpha_m-\frac{m^2}{1-\mu^2}\right)G_m = 0, \tag{94}$$

where $\alpha_m = m^2-2\Omega m/\sigma$. Since $G_m = 0$ for $\mu = \pm 1$, (94) constitutes an eigenvalue problem where α_m is the eigenvalue. Eigenfunctions may be expanded in normalized associated Legendre functions $P_n^m(\mu)$ so that

$$G_m(\mu) = \sum_{n=m}^{n_*} A_n^m P_n^m(\mu). \tag{95}$$

Substitution of (95) into (94) and use of the standard formulae,

$$(1-\mu^2)\frac{\mathrm{d}^2P_n^m}{\mathrm{d}\mu^2}-2\mu\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu}+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]P_n^m = 0,$$

$$(1-\mu^2)\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu} = (n+1)\,\epsilon_n^m P_{n-1}^m-n\epsilon_{n+1}^m P_{n+1}^m,$$

and $$\mu P_n^m = \epsilon_{n+1}^m P_{n+1}^m+\epsilon_n^m P_{n-1}^m,$$

where $\epsilon_n^m = (4n^2-1)^{\frac{1}{2}}/(n^2-m^2)^{\frac{1}{2}}$ gives, on collecting terms,

$$\sum_{n=m}^{n_*} A_n^m\{\epsilon_{n+1}^m[\alpha_m+n(1-n)]P_{n+1}^m+\epsilon_n^m[\alpha_m-(n+1)(n+2)]P_{n-1}^m\} = 0.$$

The orthogonality of P_n^m then implies a two-term recurrence relation

$$A_{s-1}^m\epsilon_s^m\Gamma_{s-2}^m+A_{s+1}^m\epsilon_{s+1}^m\Gamma_{s+2}^m = 0\quad(s = m, m+1\ldots), \tag{96}$$

where $\Gamma_s^m = m^2 - 2\Omega m/\sigma - s(s+1)$. If the function $\Gamma_r^m = 0$ for some r then it can be shown that all of the coefficients vanish except A_{r-1}^m and A_{r+1}^m giving a set of eigenfunctions $G_m^{(r)}(\mu)$ with

$$G_m^{(r)}(\mu) = A_{r-1}^m P_{r-1}^m + A_{r+1}^m P_{r+1}^m \tag{97}$$

and
$$\sigma = \frac{-2\Omega m}{r(r+1)-m^2}. \tag{98}$$

The dispersion relation (98) should be contrasted with the corresponding equation for the Rossby–Haurwitz wave, i.e.

$$\sigma = \frac{-2\Omega m}{r(r+1)}.$$

Let
$$v' \cos\phi = \mathrm{i} m V'_m(\mu)\, \mathrm{e}^{\mathrm{i}(m\lambda - \sigma t)} \tag{99}$$

and use (92) to obtain an expression for V'_m in terms of $G_m(\mu)$, then it may be shown that

$$V'_m(\mu) = \frac{1}{2\Omega a}\left[\frac{G_m}{\mu} + \frac{(1-\mu^2)}{\mu^2}\left(\frac{\sigma}{2\Omega m}\right)\frac{\mathrm{d}G_m}{\mathrm{d}\mu}\right]. \tag{100}$$

Using (96), (97), the condition $\Gamma_r^m = 0$ and the recurrence formulae for the associated Legendre function, it can be shown that

$$V'_m(\mu) = \frac{A_{r+1}^m P_r^m(\mu)}{2\Omega a \epsilon_{r+1}^m} \tag{101}$$

– exactly the same spherical harmonic form as the Rossby–Haurwitz wave though with the distorted phase speed formula (98). Considering the case $m = 1$, the angular phase speed (σ/m) is grossly in error only for the gravest planetary mode $r = 1$ for which the phase speed is -2Ω rather than the true value of $-\Omega$. For $r = 2$, σ/m is $-2\Omega/5$ rather than the true value of $-\Omega/3$, and $r = 3$ gives $-2\Omega/11$ rather than $-\Omega/6$: convergence to the true phase speed is rapid. The accuracy of the modes with $r \gg m$ is clearly related to the elongation of the implied eddy circulations in the zonal direction so that the contribution of the zonal flow components to the total kinetic energy outweighs that of the meridional components (wherein lies the main approximation in the derivation of the planetary semi-geostrophic equations). For higher zonal wavenumbers the error in the gravest meridional mode $(m = r)$ becomes relatively worse since the eddy circulations are then elongated in the meridional direction. In fact, for large $r\,(=m)$, the angular phase speed of these modes tends to $-2\Omega/r$ rather than the correct value of $-2\Omega/r^2$.

For at least two reasons this Rossby–Haurwitz wave test of the accuracy of the PSG equations is unduly severe and should not be taken on its own as a measure of the practical value of the equation set. First, the Rossby–Haurwitz wave is not an accurate model of large-scale travelling wave motion. The rate of retrogression (westward propagation relative to the airflow) is far greater than is observed, due primarily to the neglect of divergence and the associated vertical structure of real planetary Rossby waves. The total energy associated with these waves involves a potential energy component as well as the kinetic energy; their ratio is given by the Burger number defined in §2.2. Therefore, the error introduced through the neglect of $\frac{1}{2}(v\cos\phi)^2$ from the Hamiltonian should be measured against the sum of the remaining kinetic energy $(\frac{1}{2}(v\sin\phi)^2 + \frac{1}{2}u^2)$ *and* the potential energy of the wave. As shown earlier, for motions with small Burger number the entire kinetic energy may be neglected and the Phillips Type II equations obtained.

To assess the accuracy of the planetary semi-geostrophic equations in a more realistic context we look at the properties of stationary, three-dimensional planetary Rossby waves and compare PSG solutions with the corresponding primitive equation solutions.

3.2. *Stationary, baroclinic planetary waves*

Assume, for simplicity, small-amplitude sinusoidal perturbations of a basic state atmosphere in solid rotation with zonal velocity $\bar{U}\cos\phi$ (where $\bar{U}$ is constant) and with potential temperature increasing linearly with Z so that

$$\theta = \theta_0(1+\bar{B}Z),$$

where $\bar{B}$ is the static stability; the zonal wind is supported by a geopotential $\bar{\Phi}$ given by

$$\bar{U}\cos\phi = -\frac{1}{2\Omega a\sin\phi}\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\phi}. \tag{102}$$

Denoting the wave components by primes we have

$$u = \bar{U}\cos\phi + u', \quad v = v', \quad w = w', \quad \Phi = \bar{\Phi}+\Phi',$$

$$u_g = \bar{U}\cos\phi - \frac{1}{2\Omega a\sin\phi}\frac{\partial\Phi'}{\partial\phi}, \quad v_g = \frac{1}{2\Omega a\sin\phi\cos\phi}\frac{\partial\Phi'}{\partial\lambda} \tag{103}$$

$$\theta = \theta_0(1+\bar{B}Z)+\frac{\theta_0}{g}\frac{\partial\Phi'}{\partial Z}$$

(consistent with (79), and dropping the LV subscript from now on so that $\partial/\partial Z$ represents $(\partial/\partial Z)_{\mathrm{LV}}$) where the geostrophic wind and hydrostatic relations have been used. Substituting the required equations from (103) into the PSG momentum equations (77) and (78) with $\partial/\partial t \equiv 0$ and neglecting products of primed variables gives

$$2\Omega v'\sin\phi = \frac{1}{a\cos\phi}\frac{\partial\Phi'}{\partial\lambda} - \frac{\bar{U}}{2\Omega a^2\sin\phi}\frac{\partial^2\Phi'}{\partial\lambda\,\partial\phi}, \tag{104}$$

$$2\Omega u'\sin\phi = -\frac{1}{a}\frac{\partial\Phi'}{\partial\phi} - \frac{\bar{U}\tan\phi}{2\Omega a^2}\frac{\partial^2\Phi'}{\partial\lambda^2}, \tag{105}$$

where some factors of $(1\pm\bar{U}/2\Omega a)$ have been replaced with unity for convenience in view of the fact that $|\bar{U}|/2\Omega a \ll 1$ for typical atmospheric values of $\bar{U}$. The conservation of potential temperature can similarly be shown to reduce to

$$w' = -\frac{\bar{U}}{ag\bar{B}}\frac{\partial^2\Phi'}{\partial Z\,\partial\lambda}. \tag{106}$$

Equations (104)–(106) give expressions for u', v' and w' in terms of the perturbation geopotential Φ': these may be linked through the continuity equation for quasi-incompressible flow

$$\frac{1}{a\cos\phi}\left(\frac{\partial u'}{\partial\lambda}+\frac{\partial(v'\cos\phi)}{\partial\phi}\right)+\frac{\partial w'}{\partial Z} = 0. \tag{107}$$

We seek stationary planetary Rossby wave solutions with sinusoidal variation in λ and Z which tilt westwards with height. These correspond to modes which transmit wave energy upwards and would arise in problems with lower tropospheric forcing such as sinusoidal orography.

Let $\Phi' = G_m(\phi')\,\mathrm{e}^{\mathrm{i}(m\lambda+\nu z)}$ where ϕ' is the colatitude; substitute (104)–(106) into (107) and rearrange into an ordinary differential equation for $G_m(\phi')$ so that

$$\frac{\mathrm{d}^2 G_m}{\mathrm{d}\phi'^2}+\frac{2-\cos^2\phi'}{\cos\phi'\sin\phi'}\frac{\mathrm{d}G_m}{\mathrm{d}\phi'}-\left(\frac{m^2}{\sin^2\phi'}-\alpha_m\right)G_m = \hat{\lambda}\cos^2\phi' G_m, \tag{108}$$

where $\alpha_m = m^2+2\Omega a/\bar{U}$ and the eigenvalue $\hat{\lambda}$ corresponds to $(2\Omega a\nu)^2/g\bar{B}$. Upward energy propagation corresponds to $\nu^2 > 0$ – otherwise the modes are evanescent. The eigenvalue problem for $G_m(\phi')$ and $\hat{\lambda}$ requires the specification of polar boundary conditions. Orszag (1974) showed that the following natural boundary condition is appropriate:

$$\frac{\mathrm{d}^k G_m(\phi')}{\mathrm{d}\phi'^k} = 0 \quad \text{at } \phi' = 0, \pi \quad \text{for } k = 0, 1, \ldots |m|-1.$$

It is not necessary to satisfy these constraints precisely given a truncated spectral or pseudo-spectral expansion of the function G_m (Boyd 1978). All that is required is that the error in satisfying these conditions can be reduced indefinitely by taking enough terms in the expansion. The pseudo-spectral approach (using modified Fourier basis functions) recommended by Boyd (1978) was used to solve the eigenvalue problem. The modified Fourier basis functions $\Theta_j(\phi')$ are given by

$$\Theta_j(\phi') = \begin{cases} \sin\phi'\cos(j\phi') & m \text{ odd} \\ \cos(j\phi') & m \text{ even,} \end{cases}$$

and are associated with a weighting function $w_i = \delta(\phi-\phi_j')$ (δ is the Dirac delta function) with

$$\phi_j' = \frac{2\pi j}{n_*}, \quad G_m(\phi') = \sum_{n=0}^{n_*-1} a_n\,\Theta_n(\phi'), \tag{109}$$

where n_* is the number of terms in the expansion. If (108) is written as

$$\mathrm{L}(G_m) = 0,$$

where L is the implied differential operator, then the pseudo-spectral method gives n_* equations

$$\sum_{j=0}^{n_*-1}\left\{\int_0^\pi w_i\,\mathrm{L}(\Theta_j)\sin\phi'\,\mathrm{d}\phi'\right\}a_j = 0 \quad (i = 0, n_*-1) \tag{110}$$

which may be written as a matrix equation of the form

$$\boldsymbol{DA} = \hat{\lambda}\boldsymbol{EA}, \tag{111}$$

where $\boldsymbol{A}$ is the coefficient vector whose components are $a_0, a_1 \ldots a_{n_*-1}$ and $\boldsymbol{D}$ and $\boldsymbol{E}$ are two matrices. The matrix eigenvalue problem (111) was solved using a standard routine in the NAG mathematical subroutine library. $G_m(\phi')$ can be obtained by substituting the a_n values into (109) and evaluating the sum. The vertical wavelength l_z of the stationary Rossby wave can be obtained from $\hat{\lambda}$ using

$$l_z = \frac{2\pi}{\nu} = \frac{4\pi\Omega a}{(g\bar{B}\hat{\lambda})^{\frac{1}{2}}}. \tag{112}$$

Primitive equation solutions corresponding to the above PSG eigenvalue problems were obtained by the pseudo-spectral technique as used by Ahlquist (1982) for travelling planetary-scale waves. For both PSG and primitive equations the following parameter values were chosen:

$$\Omega = 7.292\times10^{-5}\ \mathrm{s^{-1}}, \quad a = 6.371\times10^{6}\ \mathrm{m}, \quad \bar{U} = 14.14\ \mathrm{m\,s^{-1}}, \quad g\bar{B} = 1\times10^{-4}\ \mathrm{s^{-1}}$$

and eigenvalues were re-expressed in terms of the corresponding vertical wavelengths. The resulting eigenfunctions were normalized so that

$$\int_0^\pi G_m^2 \sin\phi' \, \mathrm{d}\phi' = 1.$$

Figures 2 (*a*)–2 (*d*) show the gravest antisymmetric geopotential perturbations from the PSG equations (solid line) and primitive equations (dashed line) for wavenumbers 1, 3, 5 and 7 respectively. Also indicated are the corresponding vertical wavelengths with negative values representing decay lengths if the waves are evanescent. The overall comparison is excellent: only wavenumber 7 shows signs of substantial discrepancy.

3.3. *Equatorially trapped waves*

If the PSG equations are to be integrated over a spherical domain it is important to establish the characteristic behaviour of wave motions near the equator. The equatorial beta-plane analysis of Matsuno (1966) revealed that amongst other species of wave motion there were eastward-propagating Kelvin waves and westward-propagating Rossby waves (symmetric in pressure about the equator). These have non-zero pressure perturbation along the equator implying an infinite meridional geostrophic wind component. Any system of balanced equations (e.g. Salmon's variable-f equations) using the geostrophic wind approximation in the acceleration terms will fail to describe these important equatorially trapped modes. The PSG equations, however, do not suffer from this defect since the meridional acceleration term is $\sin\phi \mathrm{D}(v_g \sin\phi)/\mathrm{D}t$ which vanishes at the equator. By good fortune, the kinetic energy associated with meridional motion is $\frac{1}{2}(v_g \sin\phi)^2$ and so remains bounded at the equator. The following analytic study shows that well-behaved Rossby and Kelvin modes result, and are accurate in the long-wave limit.

The equations to be used are (91)–(93) except for the addition of a free-surface term $+c^{-2}\cos\phi\, \partial\Phi'/\partial t$ included on the left-hand side of the continuity equation (93). Φ' then represents g times the deviation of the depth of a shallow fluid from its mean value H, and $c^2 = gH$ (i.e. the usual shallow-water approximation). By assuming the same form of travelling wave as in §3.1 we obtain the meridional structure equation

$$(1-\mu^2)\frac{\mathrm{d}^2 G_m}{\mathrm{d}\mu^2} - \frac{2}{\mu}\frac{\mathrm{d}G_m}{\mathrm{d}\mu} + \left(\alpha_m - \frac{m^2}{1-\mu^2} - \mu^2\lambda^2\right) G_m = 0, \tag{113}$$

where $\lambda = 2\Omega a/c$. The first-derivative term can be removed from (113) by defining a new dependent variable $G_m^*(\mu)$ such that

$$G_m = G_m^* \frac{\mu}{(1-\mu^2)^{\frac{1}{2}}}, \tag{114}$$

which on substitution into (113) can be shown to give

$$\frac{\mathrm{d}^2 G_m^*}{\mathrm{d}\mu^2} + \left[\frac{\alpha_m}{1-\mu^2} - \frac{m^2}{(1-\mu^2)^2} + \frac{(3\mu^2-2)}{\mu^2(1-\mu^2)^2} - \frac{\mu^2\lambda^2}{1-\mu^2}\right] G_m^* = 0. \tag{115}$$

Inverse powers of $1-\mu^2$ may be expanded (for small μ) using the binomial series so that the terms $O(\mu^4)$ are neglected; this leads to

$$\frac{\mathrm{d}^2 G_m^*}{\mathrm{d}\mu^2} + \left[-\frac{2}{\mu^2} + \alpha_m - m^2 - 1 + (\alpha_m - 2m^2 - \lambda^2)\mu^2 + O(\mu^4)\right] G_m^* = 0. \tag{116}$$

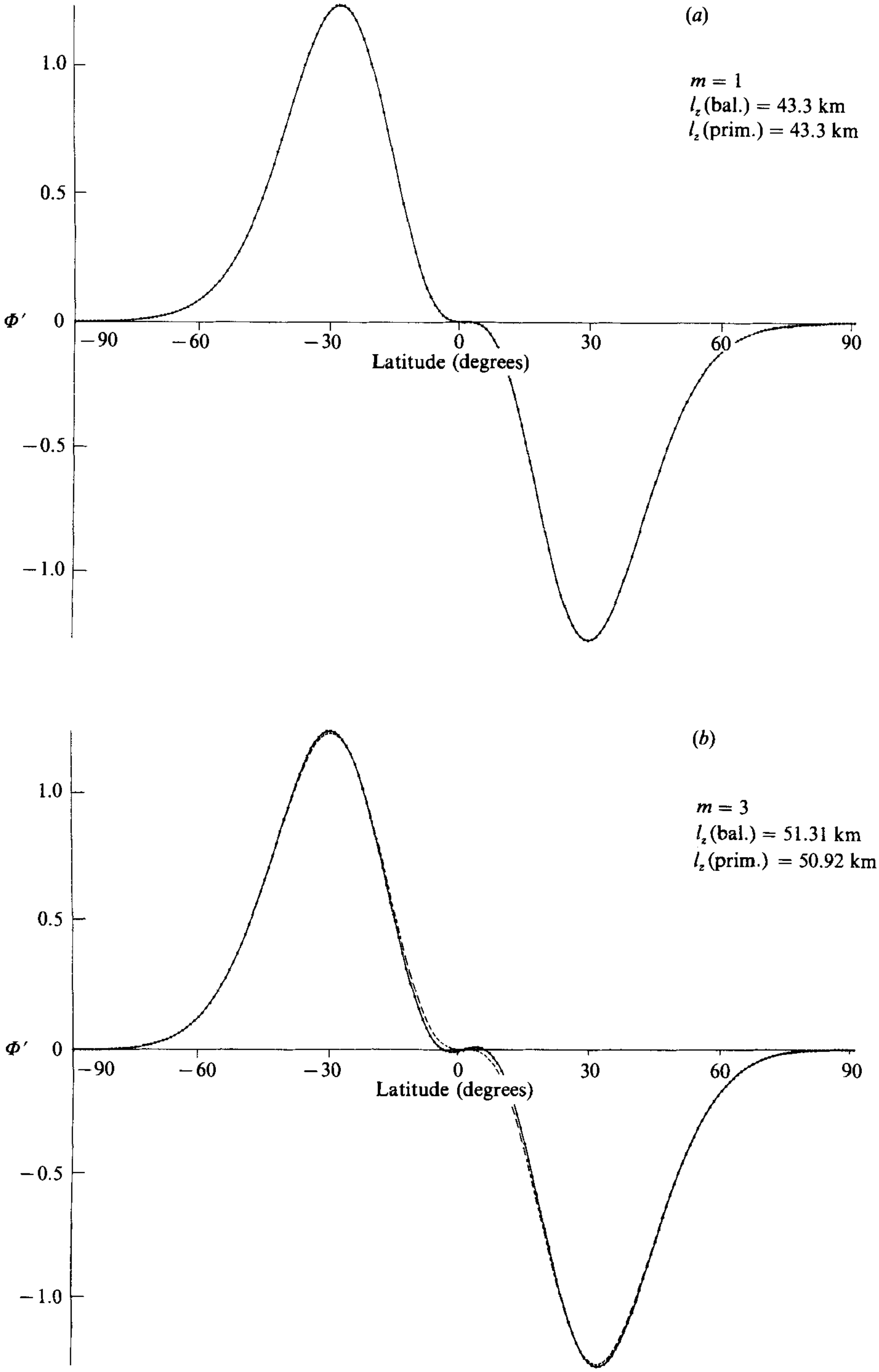

FIGURE 2(a, b). For caption see facing page.

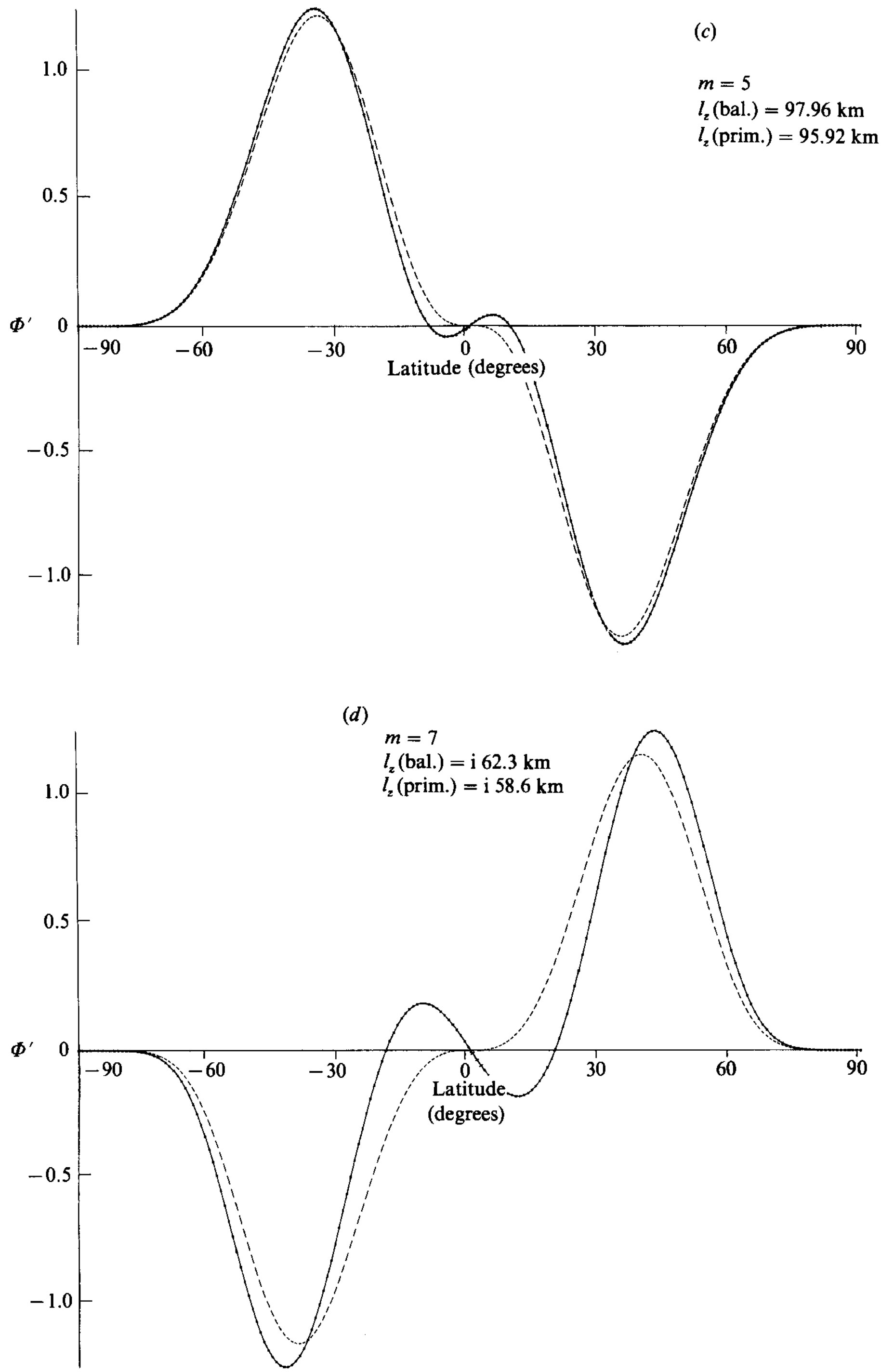

FIGURE 2(*a–d*). Gravest antisymmetric eigenfunctions $G_m(\mu)$: $m = 1$, 3, 5 and 7 respectively for PSG equations (solid line) and primitive equations (dashed line). In (*a*) these lines are indistinguishable when plotted.

Using the definition $\alpha_m = m^2 - 2\Omega m/\sigma$, (116) may be written as

$$\frac{\mathrm{d}^2 G_m^*}{\mathrm{d}\mu_*^2} + \left[4n + 2\alpha + 2 - \frac{2}{\mu_*^2} - \mu_*^2\right] G_m^* = 0, \tag{117}$$

where $\mu = \gamma\mu_*,\quad \gamma^2 = (m^2 + \lambda^2 + 2\Omega m/\sigma)^{-\frac{1}{2}},\quad \alpha = \pm\frac{3}{2}$

and
$$4n + 2\alpha + 2 = -(1 + 2\Omega m/\sigma)\gamma^2. \tag{118}$$

Equation (117) has solutions

$$G_m^*(\mu_*) = \mu_*^{\alpha+\frac{1}{2}} \exp(-\tfrac{1}{2}\mu_*^2)\, L_n^\alpha(\mu_*^2), \tag{119}$$

where L_n^α is the generalized Laguerre polynomial of order n. Equation (118) constitutes a dispersion relation ($\sigma = \sigma(m, n)$) for the two branches $\alpha = +\frac{3}{2}, -\frac{3}{2}$, which correspond to antisymmetric and symmetric modes respectively (in G_m).

Consider the case $\alpha = -\frac{3}{2}$ with $\lambda^2 \gg 1, m^2, |2\Omega m/\sigma|$; (118) reduces to

$$(4n - 1)\lambda \approx \frac{2\Omega m}{\sigma}$$

or
$$\frac{a\sigma}{m} \approx -\frac{c}{4n-1}. \tag{120}$$

For $n = 0$, the phase speed $a\sigma/m$ is approximately equal to $+c$, corresponding the eastward-propagating Kelvin wave. The $n = 1$ mode has a phase speed of $-\frac{1}{3}c$ and corresponds to the gravest westward-propagating Rossby wave. The pressure perturbation eigenfunctions are

$$G_m(\mu) = \frac{\exp(-\mu^2/2\gamma^2)}{(1-\mu^2)^{\frac{1}{2}}}$$

for the Kelvin wave ($n = 0$), and

$$G_m(\mu) = \frac{(1 + 2\mu^2/\gamma^2)}{(1-\mu^2)^{\frac{1}{2}}} \exp(-\mu^2/2\gamma^2)$$

for the gravest symmetric Rossby mode.

Both closely agree with the corresponding primitive equation eigenmodes in this limit (large λ^2). Using the values chosen for Ω and a in §3.2 and a gravity wave speed $c = 40\ \mathrm{m\,s^{-1}}, \lambda = 23$ which implies that $m^2 \ll \lambda^2$ if $m < 7$. The approximation $|2\Omega m/\sigma| \ll \lambda^2$ implies that

$$|4n - 1| \ll \lambda$$

which for these parameters is only true if $n = 0$ and 1.

By similar arguments to those given above, the antisymmetric eigenfunctions have an approximate phase speed formula

$$\frac{a\sigma}{m} \approx -\frac{c}{4n+5},$$

which holds if $\lambda^2 \gg m^2$ and $\lambda \gg 4n + 5$.

Table 1 shows values of normalized frequency (σ_*) for odd zonal wavenumbers 1–7 obtained using (118) compared to those of the usual equatorial beta-plane analysis based on the primitive equations. Three wave modes are featured: the Kelvin wave

	m	1	3	5	7
Kelvin wave	PSG	0.151	0.449	0.737	1.01
	Prim	0.147	0.442	0.737	1.03
Gravest symmetric Rossby wave	PSG	−0.0517	−0.1538	−0.252	−0.346
	Prim	−0.0474	−0.1415	−0.211	−0.264
Gravest antisymmetric Rossby wave	PSG	−0.0326	−0.0969	−0.159	−0.217
	Prim	−0.0316	−0.0860	−0.134	−0.173

TABLE 1. Normalized frequency σ_*

$(\alpha = -\frac{3}{2}, n = 0)$, and the gravest symmetric $(\alpha = -\frac{3}{2}, n = 1)$ and antisymmetric $(\alpha = +\frac{3}{2}, n = 0)$ Rossby waves. The normalized frequency σ_* is defined by

$$\sigma_* = \frac{\sigma}{2\Omega}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}$$

to conform with the scaling adopted in Matsuno's equatorial beta-plane analysis and that in common usage (e.g. Gill 1982). Further comment on these frequencies will be left until the next section.

For the antisymmetric modes, the perturbation pressure varies as μ^3 about the equator so that $u_g = v_g = u' = 0$ and v' is finite at $\mu = 0$. Using (119) in the zonal momentum equation it can be shown, after arduous algebraic manipulation, that v' varies as μ at the equator. Also $u' = u_g$ and

$$\frac{\partial u_g}{\partial t} + 2\Omega\mu v_g = 0 \quad \text{at} \quad \mu = 0.$$

Only v_g becomes infinite at $\mu = 0$ and this has no adverse effect in the PSG equations since v_g is always associated with a factor $\sin\phi$ in the momentum equations.

4. Discussion

In §2 it was shown that by approximating Hamilton's principle for a perfect fluid in a way that results in new canonical coordinates, a simple set of filtered equations of motion – consistent with the principle – can be derived. These equations automatically have analogues of the conservation properties of the unapproximated equations such as global energy conservation and Lagrangian conservation of potential vorticity. It was also shown that the Burger equations (Phillips quasi-geostrophic type II) are obtained if the kinetic energy is omitted from the Hamiltonian. X and Y are then the canonical coordinates and the equations of motion are (rewriting (33) and (34)) simply

$$\frac{\partial Y}{\partial \tau} = \frac{1}{2\Omega}\frac{\partial \Phi}{\partial X}, \quad \frac{\partial X}{\partial \tau} = -\frac{1}{2\Omega}\frac{\partial \Phi}{\partial Y}$$

i.e. the geostrophic wind relation. By choosing canonical coordinates $M(=2\Omega x + v)$ and $N(=2\Omega y - u)$, and neglecting both a term representing the ratio of the Lagrangian rate of turning of the wind direction to 2Ω and the contribution of the axial motion (parallel to $\boldsymbol{\Omega}$) to the kinetic energy, the semi-geostrophic evolution equations

$$\frac{\partial N}{\partial \tau} = \frac{\partial \Phi}{\partial X}, \quad \frac{\partial M}{\partial \tau} = -\frac{\partial \Phi}{\partial Y}$$

are obtained ((68) and (69)). In the f-plane theory, the latter assumption is extremely accurate: for planetary flow it entails the neglect of a significant amount of kinetic energy depending on the degree of zonality of the motion. Large-scale planetary motion in the terrestrial atmosphere is highly anisotropic with zonal velocity perturbations dominating meridional flow perturbations (Charney 1971; Shepherd 1987). Atmospheric flow on Jupiter and Saturn is even more highly biassed towards zonality (Ingersoll *et al.* 1979). Rhines (1975) has demonstrated that zonality is a natural state towards which 'turbulent' two-dimensional flows should migrate in the presence of a latitudinal gradient in the Coriolis parameter. It seems fitting, therefore, that the planetary semi-geostrophic set proposed here becomes highly accurate in this limit.

Consider the following alternative viewpoint regarding the motivation for the PSG set. Any rectilinear flow in an f-plane system can be set up in exact balance with a certain pressure and density field. We can regard such a straight flow as being *the* natural balanced state: this underlies f-plane semi-geostrophic theory. In a spherical rotating system, the simplest natural form of balanced motion is pure zonal flow which, apart from a small centrifugal term, is the limit in which the PSG equations become exact. To this extent, the PSG equations may be viewed as a logical extension of f-plane semi-geostrophic theory.

The linearized examples shown in §3 confirm the accuracy of the PSG set for zonally extended non-divergent barotropic motion; baroclinic, planetary waves with grave meridional structure and zonal wavenumbers up to 7, and for equatorial Rossby and Kelvin modes. Since real planetary wave motions are associated with potential as well as kinetic energy, the Rossby–Haurwitz wave limit considered in §3.1 is not a fair test of the usefulness of the PSG set. Most of the energy in atmospheric flow is contained in the zonal and low-wavenumber components (m = 1–4). An observational study by Shepherd (1987) shows that much of this low-wavenumber energy projects onto zonally anisotropic modes, particularly for stationary waves. These modes are also characterized by small Burger number so that the neglect of the axial component of velocity in the Hamiltonian causes minimal degradation of the accuracy. This is borne out by the analysis of §3.2.

Perhaps most surprising of all is the success of the PSG set in representing the meteorologically important equatorially trapped modes. For zonal wavenumbers up to 7 table 1 shows that the frequency error for Kelvin waves is at most 3%. In other balanced sets based on the geostrophic wind assumption these modes must be excluded because $v_g \to \infty$ at the equator. Even the commonly used linear balance equations have no Kelvin wave mode – a deficiency identified by Moura (1976). The accuracy of the PSG Kelvin mode is, of course, due to its very small meridional velocity component (zero in the equatorial beta-plane analysis) implying little loss of accuracy in neglecting the axial component of velocity in the Hamiltonian. Also, since air parcels almost follow latitude circles the Lagrangian rate of turning of the wind is very small. However, condition B (64) requires that the Lagrangian rate of turning of the geostrophic wind also be small compared with 2Ω: it is this constraint that limits the accuracy of the Kelvin wave for short waves.

The equatorial Rossby waves are less accurate in general with, for $m = 1$, a frequency error of 9% for the gravest symmetric mode and 3% for the antisymmetric mode; at $m = 7$ these errors rise to 27% and 23% respectively. The above figures for Kelvin and Rossby mode frequencies are based on an equivalent depth corresponding to a gravity wave speed of 40 m s^{-1}. For smaller equivalent depths and gravity wave speeds, the equations are more accurate: $c = 40$ m s^{-1} is generally regarded as being typical of the first baroclinic mode of the tropical atmosphere.

Salmon (1985) has derived a different set of semi-geostrophic equations valid for variable Coriolis parameter (f) by considering a shallow rotating layer of inviscid, homogeneous fluid in a Cartesian system with $f(x,y)$. Although convenient, the device of variable f in a Cartesian system is rather unnatural since it ignores important metric factors that arise, for instance, if x and y are Mercator coordinates. The derivation of the PSG equations in §2.3 does not assume any particular geometry of the underlying solid planet as is implicit in the choice of $f(x,y)$ in Salmon's approach. On the other hand, apart from the tropics, Salmon's system of equations is likely to be more accurate since the kinetic energy in the Hamiltonian is approximated by the total geostrophic kinetic energy whereas only part of this is retained in the PSG formulation of Hamilton's principle.

It should be noted that the f-plane limit of the PSG equations can only be considered to arise from making the tangent plane approximation at the (rotational) pole of the spherical system. This limit is not the same for tangent plane approximations at other points on the sphere owing to the omission of the axial velocity contribution to the kinetic energy. The usual procedure is to set up the Cartesian system with z-axis oriented in the local vertical direction and to ignore the horizontal component of $\boldsymbol{\Omega}$.

Since the Cartesian form of the PSG equations is exactly the same form as the f-plane semi-geostrophic equations, they may be solved using exactly the same techniques (Schubert 1985). It would be interesting to extend the baroclinic instability studies of Hoskins & West (1979) and Hoskins & Heckley (1981) to flows with a background potential vorticity gradient using the PSG equations in geostrophic momentum coordinate form. The equations could also be used in the study of large-amplitude planetary Rossby waves in the middle atmosphere, for which the Burger number would be small.

The time-averaged state of the terrestrial atmosphere is dominated by zonal flow and low zonal wavenumbers ($m = 1$, 2 and 3). The PSG equations could therefore form the basis of a low-order climate model of the type used by Shutts (1983) and White & Green (1982) where the time-averaged motion is represented explicitly and the dynamical effects of transient baroclinic instabilities are parametrized in terms of the transfer of potential vorticity. The existence of a diagnostic relation between the potential vorticity and Φ_* (e.g. (90)) is crucial to this type of parametrized climate model.

Finally, the PSG set could be used to study the dynamics of the intertropical convergence zone since the phenomenon could be considered to be quasi-zonal. The geostrophic momentum coordinate transformation could be used to obtain the coordinate stretching property used with such success in mid-latitude frontogenesis studies.

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